The Origin and Aftermath of the Pythagorean Theorem

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One of the few pieces of math which nearly everyone remembers from geometry class is the Pythagorean theorem: the squared length of the hypotenuse is equal to the sum of the squared lengths of the legs. If we let $a, b,$ and c denote the lengths of the two shorter sides and the hypotenuse of a triangle respectively, it can be simply expressed as the following equation:

$$a^2 + b^2 = c^2.$$

This supposedly Pythagorean result is among the simplest and most well-known theorems in mathematics, specifically euclidean geometry, yet one would make a grave mistake to deem it as therefore trivial. The theorem is both mathematically and historically rich; it can be generalized to a great deal of interesting math—to non-euclidean or differential geometries for example—and its history far predates the Greek philosopher it is named after.

Evidence of civilizations' knowledge of the theorem can be traced back to ancient Mesopotamia ca. 1800 BC where integer solutions to the theorem, known as Pythagorean triples, are inscribed in Sumerian clay tablets and were used anywhere where the deployment of right angles was desired—e.g., land surveying and construction [1].

While the application of a concept can be very powerful—even if not totally grasped—further insights, and even more powerful tools which come from those insights, are usually only possible after preceding concepts are completely understood. One rigorous metric for gauging understanding in mathematics is to write a proof. Much of Pythagoras's legacy rests on him being the first person to have proven what we today call the Pythagorean theorem. Yet according to recent scholarship, that legacy is unsubstantiated; the Stanford Encyclopedia of Philosophy asserts 'it is not likely that he proved the theorem' [2].

So what *can* we be certain of regarding Pythagoras and his accomplishments? Unfortunately, it seems to be not much; not a single primary source pertaining to Pythagoras has been found, which is unsurprising as he is also believed to have not written any books or recorded any of his work. All we know with near certainty is that he was born on the island of Samos ca. 570 BC, perhaps traveling to Egypt or other regions east of Greece for a few decades, before emigrating to the city of Croton in southern Italy ca. 530 BC where he did most of his philosophizing [2].

It is in Croton where Pythagoras garnered fame and infamy alike as he promulgated his beliefs and amassed disciples. Following Pythagoras's death, a group named the Divine Brotherhood of Pythagoras emerged—also referred to as the Pythagoreans and which will hereafter be abbreviated as the D.B.P. It is likely that nearly all of the mathematical developments Pythagoras is credited with actually came from various members of this secretive group. There is also doubt whether any of the D.B.P's members actually interacted with Pythagoras. Aristotle often referred to the Pythagoreans as 'so-called,' which 'calls into question the actual connection between these thinkers and Pythagoras himself' [3]. What he taught his original followers, like most of the details surrounding Pythagoras's life, is also shrouded in mystery, but it is generally believed that he prescribed his acolytes to follow a strict daily routine, investigate numerical relationships to attain insight into the cosmos, and espouse metempsychosis—the transmigration of the soul after death into another body of the same or different species. (Another supposed and amusing tenet the Pythagoreans held was to never eat beans because they believed that legumes had souls of their own.)

Regardless of whether Pythagoras was the mathematical prodigy he is widely believed to be today, it is about time that we gave some proofs of the theorem he is credited with. The Pythagorean theorem is famous in mathematics in that it is probably the theorem with the single greatest number of proofs. The first of three that we will cover is below. It should be noted that all of these proofs require axioms of Euclidean Geometry—one of the most significant ones being that for any line and any point not on that line, there is a unique line which passes through that point and is parallel to other line.

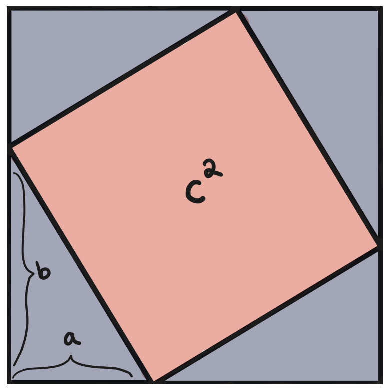


Figure 1: The four blue-grey right triangles are congruent and have legs of length a and b and hypotenuse of length c.

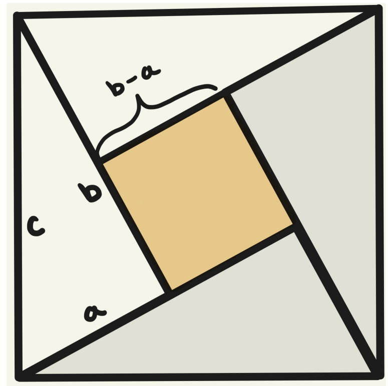
Proof. Observe that in figure 1 the area of the square which takes up the entire figure is $(a+b)^2$, and the area of the red square in the center is c^2, as labeled. Therefore:

$$[\text{Area of the Entire Square}] = [\text{Area of the Red Square}] + 4 \cdot [\text{Area of a Blue-Grey Right Triangle}]$$

$$\begin{aligned} (a+b)^2 &= c^2 + 4\left(\frac{1}{2}ab\right) \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 &= c^2 \qquad \blacksquare \end{aligned}$$

(The black square written directly above does nothing other than indicate the end of the proof.)

Here's another geometric proof which becomes clear with some algebraic simplification.



Proof. Observe that the area of the entire square is c^2 and that area is itself made up of the areas of four congruent right triangles with legs of length a and b and the area of the smaller yellow square in the center which is $(b-a)^2$. Expressing these observations as an equation we have:

[Area of the Entire Square] = [Area of the Yellow Square] + 4 · [Area of Smaller Right Triangle]

$$c^2 = (b-a)^2 + 4\big(\frac{1}{2}ab\big)$$
$$c^2 = b^2 - 2ab + a^2 + 2ab$$
$$c^2 = a^2 + b^2 \qquad \blacksquare$$

While these proofs are clear once you abstract various lengths into variables and perform algebraic simplifications, it would be difficult to intuit that they were true from just looking at the diagrams. Members of the D.B.P didn't express numbers as variables (that practice only began around the time of René Descartes ca. the 17th century), so it is natural to prefer a proof which uses the methods only then available to the ancient greeks. One such proof, which is entirely visual, can be found when we re-arrange the objects in figure 1.

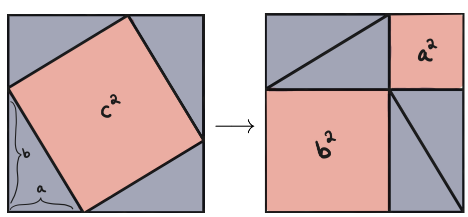


Figure 2: A rearrangement of figure 1.

We should observe from figure 2 that the red-colored areas in the two figures are the same, as we only re-arranged where the four blue-grey triangles were, which directly gives us that $a^2 + b^2 = c^2$.

One might expect that the D.B.P celebrated the proof of the Pythagorean theorem, yet the opposite was true. Knowledge of the theorem brought with it an understanding of what the Greeks then called '*arratos*' (meaning 'not-having-a-ratio') if one simply investigated the length of a square's diagonal. By the Pythagorean theorem, the diagonal of a Unit Square is a number such that its square is 2. The D.B.P found that 'no matter how small a unit of measure is used, the side of a Unit Square is *incommensurable* with the diagonal' [4]. This was profoundly disturbing to the Pythagoreans because it 'shattered their deeply held belief in the supremacy of whole numbers as the underlying principle of the universe' [5]; in failing to express something as quotidian as the diagonal of a Unit Square as a ratio of whole numbers, it is no exaggeration to say that the D.B.P's conception of reality was upended. Given this fact, one might have expected the D.B.P to simply ignore these quantities (which they may have tried for some time), yet proving the number whos square is two (which will hereafter be denoted $\sqrt{2}$) to be irrational was surprisingly simple, and it was impossible for the D.B.P to ignore it forever if they had any sense of mathematical integrity.

To prove the irrationality of the $\sqrt{2}$, one need only assume the opposite—namely, that the $\sqrt{2}$ *is rational*—then demonstrate how that assumption leads to a contradiction (a statement which is always false). After all, a number is either rational or irrational—so if it being rational leads to a contradiction, then the other possibility—that it is irrational—must be true. N.b. that we are employing a form of proof called a *proof by contradiction*. (Be apprised though, that there exist simple unresolvable contradictions like 'this sentence is a lie'.)

Before jumping into the proof, let's remind ourselves that a number is rational if it can be expressed as the quotient of two integers. But following from that definition, we also know that every rational number must have a fully simplified representation.

Consider $\frac{126}{3003}$. This is indeed a rational number, but if we write out the prime factorization of the numerator and denominator, we see it is $\frac{2 \cdot 3^2 \cdot 7}{3 \cdot 7 \cdot 11 \cdot 13}$ which can at most be simplified to $\frac{6}{143}$ because the numerator and denominator share no factors. Every genuine rational number then has a fully simplified form, and this will be a crucial requirement to consider when proving that the $\sqrt{2}$ is irrational.

Let's go ahead and do that now.

$$\text{Let the } \sqrt{2} = \frac{a}{b}, \quad \text{where } a \text{ and } b \text{ are integers and they share no factors} \tag{1}$$

Squaring both sides leaves us with:

$$(\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 \implies 2 = \frac{a^2}{b^2} \implies 2b^2 = a^2 \tag{2}$$

Now notice that a^2 is an integer multiple of 2, so it is even, since b^2 is just an integer times itself which will also be an integer. (Remember that what it means for an integer n to be even is if for some other integer k, $n = 2k$). Now if a^2 is even, what does that say about a? A simple way to find out is by considering the square of an arbitrary even or odd number algebraically.

Let $o = 2k + 1$ for some integer k be an odd number. Then

$$\begin{aligned} o^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

We see then that o^2 is of the form $2m + 1$ (if we let $m = 2k^2 + 2k$), so it is also odd.

Let $e = 2k$ for some integer k be an even number. We find that e^2 is also even because

$$\begin{aligned} e^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

This is not surprising because intuitively we know that if we take a number which has 2 in its prime factorization and square it, we will attain a number with two copies of 2 in its factorization, so it also is even. But if we take a number which does not have 2 in its factorization (an odd number) and square it, we will not magically attain any copies of 2 in the number squared, so it will remain odd.

Therefore, we know a is even and thus $a^2 = (2k)^2 = 4k^2$ for some integer k. So plugging this into equation (2), we have

$$2b^2 = a^2 \implies 2b^2 = 4k^2 \implies b^2 = 2k^2.$$

And so, using the same reasoning that we did for a^2 and a being even, b^2 and b must also be even. But if both a and b are even, i.e., $a = 2\ell$ and $b = 2k$ for some integers ℓ and k, then

$$\sqrt{2} = \frac{a}{b} = \frac{2\ell}{2k} = \frac{\ell}{k}$$

which means that $\frac{a}{b}$ was not in its fully simplified form. A mathematical alarm should be going off now; we started our proof knowing that $\frac{a}{b} = \sqrt{2}$ was simplified, but that implied it was *not* actually simplified. This is precisely the contradiction we sought. And in reaching this statement which can never be true, that $\frac{a}{b}$ is both simplified and not simplified; we prove that the assumption must be false, that is, that the $\sqrt{2}$ must be *irrational*. $\blacksquare$

While this is a correct proof, it is also much longer than what one would often write or would be necessary. The goal of this proof was, yes, to show that the $\sqrt{2}$ is irrational, but it was also to explain and justify all steps for an audience which may not be familiar with most of mathematics. Often, the extent to which various steps are justified in a proof varies depending on how much your audience already knows or needs explained to convince them the validity of your claim (though you would likely never actually write as much as was written for this proof other than for pedagogical purposes). Here is the same valid proof that the $\sqrt{2}$ is irrational distilled.

Proof.
Let $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $(p, q) = 1$.
Then $2q^2 = p^2$ so p^2 and p are even. Thus $\exists k \in \mathbb{Z}$ s.t. $p = 2k$ so
$2q^2 = 4k^2 \implies q^2 = 2k^2$.
Therefore q^2 and q are also even, so (p, q) is at least 2 which is $\neq 1$. $\Rightarrow\!\Leftarrow$ $\blacksquare$

A proof of this form is usually preferred because only the most salient steps are described, so the entire proof is more concise to someone familiar with math at the risk of seeming more esoteric to one less acquainted with mathematical notation or concepts.

References

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