Solving Inhomogeneous 2nd Order Differential Equations Using Fourier Series

Charles Kolozsvary M597F

May 21, 2023

After taking an introductory ODEs course, one would find that solving 2nd order *homogeneous* ODEs is rather simple. For a DE (differential equation) of the form

$$y'' + by' + cy = 0,$$

we can find the solution to the characteristic equation $r^2 + br + c = 0$, which gives

$$r = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \alpha \pm \beta$$

by the quadratic formula. Depending on the discriminant $b^2 - 4c$, the solutions to the characteristic equation ($\alpha + \beta$ and $\alpha - \beta$) fall into three different categories, and correspondingly, there emerge three kinds of solutions to the DE.

$$y(x) = \begin{cases} e^{\alpha x}\Big(c_1 \cos(-i\beta x) + c_2 \sin(-i\beta x)\Big) & \text{if } b^2 < 4c \\ c_1 e^{\alpha x} + c_2 x e^{\alpha x} & \text{if } b^2 = 4c \\ c_1 e^{(\alpha+\beta)x} + c_2 e^{(\alpha-\beta)x} & \text{if } b^2 > 4c \end{cases}$$

Note: in the first case, if $b^2 < 4c$, then $-i\beta \in \mathbb{R}$.

1 Inhomogenous 2nd Order Differential Equations

But how would we solve a DE if the right hand side was not zero (if the DE was *inhomogeneous*)? For example, if we had a DE of the form

$$y'' + by' + cy = f.$$

In such a case, one would need to apply a different method to find a solution—say, applying a Laplace transform, though, for an arbitrary forcing function f this could prove difficult. Yet if f is periodic, then it is relatively easy to find a solution using Fourier series.

1.1 Fourier Series

Recall that for a periodic function $f(x)$ over the interval $[0, 2L]$,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

is the Fourier series of $f(x)$, where

$$a_0 = \frac{1}{2L}\int_0^{2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Suppose that we want to solve a DE of the form

$$y'' + Ey' + Fy = f$$

where f is periodic over $[0, 2\pi]$. Then a good guess for the particular solution (or the *inhomogeneous* solution) to the DE would be

$$y_p(x) = a_0^\star + \sum_{n=1}^{\infty} a_n^\star \cos(nx) + \sum_{n=1}^{\infty} b_n^\star \sin(nx)$$

since f is itself of that form. The coefficients $a_0^\star$, $a_n^\star$, and $b_n^\star$ are different from the Fourier series of f and are determined by plugging in the guess $y_p(x)$ back into the original DE.

Other than the particular solution y_p, there also exists the *homogeneous* solution, y_h, which can be easily found from what we mentioned earlier. With both y_p and y_h in hand, we can find the overall solution to the DE:

$$y(x) = y_p(x) + y_h(x).$$

To better grasp how one might solve these *inhomogeneous* 2nd order DEs, we will consider some examples and their solutions using Fourier series, then compare those solutions with the numerical ones given by the ODE solver `odeint` which is a part of the `scipy.integrate` Python library.

1.2 Integral Identities

Recall the following integral identities, for $n, m \in \mathbb{N}_{>0}$, which are frequently used when computing Fourier series.

$$\int_0^{2\pi} \sin(nx)dx = 0 \qquad \int_0^{2\pi} \sin(nx)\sin(mx)dx = \pi\delta_{n,m}$$

$$\int_0^{2\pi} \cos(nx)dx = 0 \qquad \int_0^{2\pi} \cos(nx)\cos(mx)dx = \pi\delta_{n,m}$$

$$\int_0^{2\pi} \sin(nx)\cos(nx)dx = 0$$

Where $\delta_{n,m}$ is the Kronecker delta function

$$\delta_{n,m} = \begin{cases} 1 & \text{for } n = m \\ 0 & \text{for } n \neq m \end{cases}$$

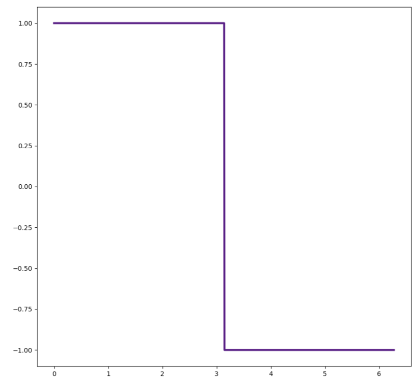
2 Discontinuous Example: Square Wave as Forcing Function

Consider the differential equation

$$y'' + 2y' + 5y = f_{sq}$$

where

$$f_{sq}(x) = \begin{cases} 1 & \text{if } x < \pi \pmod{2\pi} \\ -1 & \text{if } x \geq \pi \pmod{2\pi}. \end{cases}$$



and initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

2.1 Computing Fourier Series of f_{sq}

With f_{sq} periodic with period 2π, we can find its Fourier series by computing

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f_{sq}(x)dx = \frac{1}{2\pi}\left[\int_0^{\pi} dx - \int_{\pi}^{2\pi} dx\right] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f_{sq}(x)\cos(nx)dx = \frac{1}{\pi}\left[\int_0^{\pi} \cos(nx)dx - \int_{\pi}^{2\pi} \cos(nx)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\int_0^{\pi} n\cos(nx)dx - \frac{1}{n}\int_{\pi}^{2\pi} n\cos(nx)dx\right] = \frac{1}{n\pi}\left[\sin(nx)\Big|_0^{\pi} - \sin(nx)\Big|_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f_{sq}(x)\sin(nx)dx = \frac{1}{\pi}\left[\int_0^{\pi} \sin(nx)dx - \int_{\pi}^{2\pi} \sin(nx)dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\int_0^{\pi} n\sin(nx)dx - \frac{1}{n}\int_{\pi}^{2\pi} n\sin(nx)dx\right] = \frac{1}{n\pi}\left[-\cos(nx)\Big|_0^{\pi} + \cos(nx)\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{n\pi}\left[-\cos(n\pi) - -\cos(0) + \cos(2n\pi) - \cos(n\pi)\right] = \frac{1}{n\pi}\left[2 - 2\cos(n\pi)\right]$$

Recall that $\cos(n\pi) = 1$ for even n, and $\cos(n\pi) = -1$ for odd n. Thus,

$$b_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ odd} \end{cases}$$

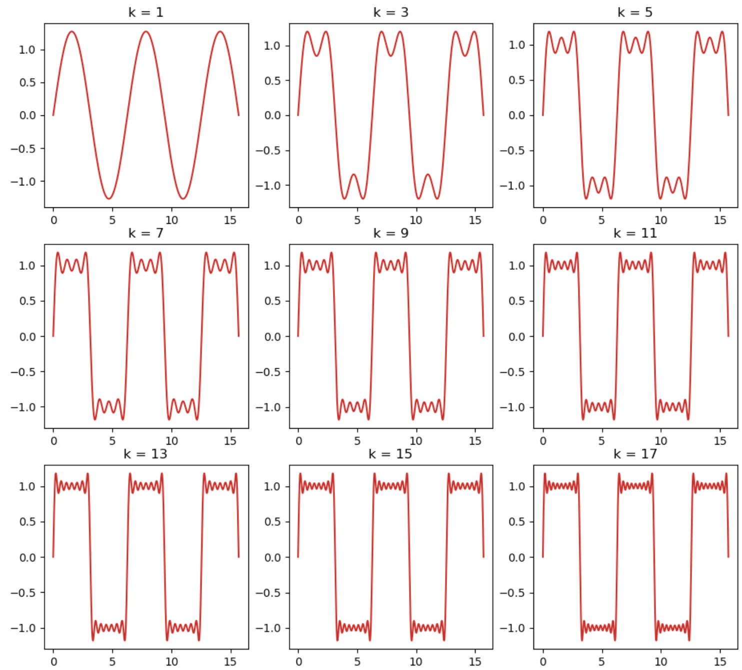


Figure 1: Plotting the Fourier series of the f_{sq} for finitely many terms k.

Or equivalently,

$$b_n = \frac{2(1-(-1)^n)}{n\pi}$$

Therefore

$$f_{sq}(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

We could have used the fact that if f is an odd function, then the coefficients of its Fourier series are only those for the sin terms, i.e., that $a_n = 0$ for $n \geq 0$, but it was good to verify this fact for f_{sq} (yet we will keep this in mind for the next example).

2.2 Computing y_p

We make the guess that the particular solution to the DE is

$$y_p = a_0^\star + \sum_{n=1}^{\infty} a_n^\star \cos(nx) + \sum_{n=1}^{\infty} b_n^\star \sin(nx).$$

Therefore

$$y_p' = \sum_{n=1}^{\infty} -na_n^\star \sin(nx) + \sum_{n=1}^{\infty} nb_n^\star \cos(nx)$$

$$y_p'' = \sum_{n=1}^{\infty} -n^2 a_n^\star \cos(nx) + \sum_{n=1}^{\infty} -n^2 b_n^\star \sin(nx)$$

Plugging y_p'', y_p', and y_p into our original DE $y'' + 2y' + 5y = f_{sq}$ yields

$$1\left[\sum_{n=1}^{\infty} -n^2 a_n^{\star} \cos(nx) + \sum_{n=1}^{\infty} -n^2 b_n^{\star} \sin(nx)\right] +$$

$$2\left[\sum_{n=1}^{\infty} -n a_n^{\star} \sin(nx) + \sum_{n=1}^{\infty} n b_n^{\star} \cos(nx)\right] +$$

$$5\left[a_0^{\star} + \sum_{n=1}^{\infty} a_n^{\star} \cos(nx) + \sum_{n=1}^{\infty} b_n^{\star} \sin(nx)\right] = f_{sq}$$

Let

$$\mathcal{S}\Big(z\Big) = \sum_{n=1}^{\infty} z \sin(nx), \quad \mathcal{C}\Big(z\Big) = \sum_{n=1}^{\infty} z \cos(nx).$$

Then

$$f_{sq} = \mathcal{S}\Big(b_n\Big) = \mathcal{S}\left(\frac{(2-(-2)^n)}{n\pi}\right) \implies$$

$$\mathcal{C}\Big(-n^2 a_n^{\star}\Big) + \mathcal{S}\Big(-n^2 b_n^{\star}\Big) + \mathcal{S}\Big(-2n a_n^{\star}\Big) + \mathcal{C}\Big(2n b_n^{\star}\Big) + 5a_0^{\star} + \mathcal{C}\Big(5a_n^{\star}\Big) + \mathcal{S}\Big(5b_n^{\star}\Big) = \mathcal{S}\left(\frac{(2-(-2)^n)}{n\pi}\right)$$

$$\mathcal{C}\Big((5-n^2)a_n^{\star} + 2n b_n^{\star}\Big) + \mathcal{S}\Big((5-n^2)b_n^{\star} - 2n a_n^{\star}\Big) + 5a_0^{\star} = \mathcal{C}\Big(0\Big) + \mathcal{S}\left(\frac{(2-(-2)^n)}{n\pi}\right) + 0$$

By matching coefficients, we see that

$$0 = 5a_0^{\star} \implies a_0^{\star} = 0$$

$$0 = (5-n^2)a_n^{\star} + 2n b_n^{\star}$$

$$\frac{(2-(-2)^n)}{n\pi} = (5-n^2)b_n^{\star} - 2n a_n^{\star}.$$

Before we finish this computation, let's find $a_n^{\star}$ and $b_n^{\star}$ based on general coefficients of y' and y in the DE—this will save us time later.

2.3 Interlude: Computing $a_n^{\star}$ and $b_n^{\star}$ in y_p for General Coefficients

Based on the partial work just completed, we notice that if we consider the 2nd order *inhomogeneous* DE

$$y'' + Ey' + Fy = g,$$

where g has period 2π and is odd, therefore having Fourier series

$$\sum_{n=1}^{\infty} b_n \sin(nx),$$

then to find $a_0^\star$, $a_n^\star$, and $b_n^\star$ for y_p, we need only solve the system of equations

$$Fa_0^\star = 0$$

$$(F - n^2)a_n^\star + Enb_n^\star = 0 \qquad (1)$$

$$(F - n^2)b_n^\star - Ena_n^\star = b_n \qquad (2)$$

From these systems of equations we find

$$a_n^\star = \frac{-Enb_n^\star}{(F - n^2)} \qquad \text{from (1)}$$

$$\text{into (2)} \implies (F - n^2)b_n^\star - En\left(\frac{-En}{(F - n^2)}b_n^\star\right) = b_n$$

$$b_n^\star\left((F - n^2) + \frac{E^2n^2}{(F - n^2)}\right) = b_n^\star\left(\frac{(F - n^2)^2 + E^2n^2}{(F - n^2)}\right) = b_n$$

$$b_n^\star = \frac{(F - n^2)b_n}{(F - n^2)^2 + E^2n^2}$$

$$a_n^\star = \frac{-En}{(F - n^2)}b_n^\star = \frac{-En}{(F - n^2)}\left(\frac{(F - n^2)b_n}{(F - n^2)^2 + E^2n^2}\right)$$

$$a_n^\star = \frac{-Enb_n}{(F - n^2)^2 + E^2n^2}$$

2.4 Return to Example 1

Here our DE is

$$y'' + 2y' + 5 = f_{sq}$$

and the Fourier series of f_{sq} is

$$\sum_{n=1}^{\infty} b_n \sin(nx)$$

where

$$b_n = \frac{(2 - (-2)^n)}{n\pi}$$

Therefore our $a_n^\star$ and $b_n^\star$ for y_p are

$$a_n^\star = \frac{-2b_n}{(5 - n^2)^2 + 4n^2}$$

$$b_n^\star = \frac{(5 - n^2)b_n}{(5 - n^2)^2 + 4n^2}$$

since $E = 2$ and $F = 5$.

2.5 Computing y_h

By what we mentioned at the very start, y_h is simply

$$e^{-x}\Big(c_1\cos(2x)+c_2\sin(2x)\Big),$$

since

$$\frac{-2\pm\sqrt{2^2-4\cdot 5}}{2}=-1\pm 2i.$$

2.6 Initial Conditions

Our overall solution is to this DE is

$$y = y_p + y_h$$

$$y(x)=\sum_{n=1}^{\infty}a_n^\star\cos(nx)+\sum_{n=1}^{\infty}b_n^\star\sin(nx)+e^{-x}\Big(c_1\cos(2x)+c_2\sin(2x)\Big)$$

Therefore to find c_1 and c_2 we use our initial conditions $y(0)=1$ and $y'(0)=0$.

$$y(0)=\sum_{n=1}^{\infty}a_n^\star\cos(0)+\sum_{n=1}^{\infty}b_n^\star\sin(0)+e^{0}\Big(c_1\cos(0)+c_2\sin(0)\Big)$$

$$=\sum_{n=1}^{\infty}a_n^\star+c_1=1$$

$$\implies c_1=1-\sum_{n=1}^{\infty}a_n^\star$$

$$y'(0)=\sum_{n=1}^{\infty}-na_n^\star\sin(0)+\sum_{n=1}^{\infty}nb_n^\star\cos(0)+-e^{0}\Big(c_1\cos(0)+c_2\sin(0)\Big)+e^{0}\Big(-2c_1\sin(0)+2c_2\cos(0)\Big)$$

$$=\sum_{n=1}^{\infty}nb_n^\star-c_1+2c_2=0$$

$$\implies c_2=\frac{1}{2}\Big(c_1-\sum_{n=1}^{\infty}nb_n^\star\Big)$$

Note

$$\sum_{n=1}^{10000}a_n^\star\approx-0.1834\qquad\text{and}\qquad\sum_{n=1}^{10000}nb_n^\star\approx 0,$$

which will be the values we use in our solution.

2.7 Plotting Solutions of $y''+2y'+5y=f_{sq}$

Using all the information we've now found, we can finally plot our solution and the solution found by `odeint`. Seeing as we cannot compute an infinite sum required for y_p, we will instead compute

$$y_p(x,k)=\sum_{n=1}^{k}a_n^\star\cos(nx)+\sum_{n=1}^{k}b_n^\star\sin(nx)$$

for increasing $k \in \mathbb{N}$.

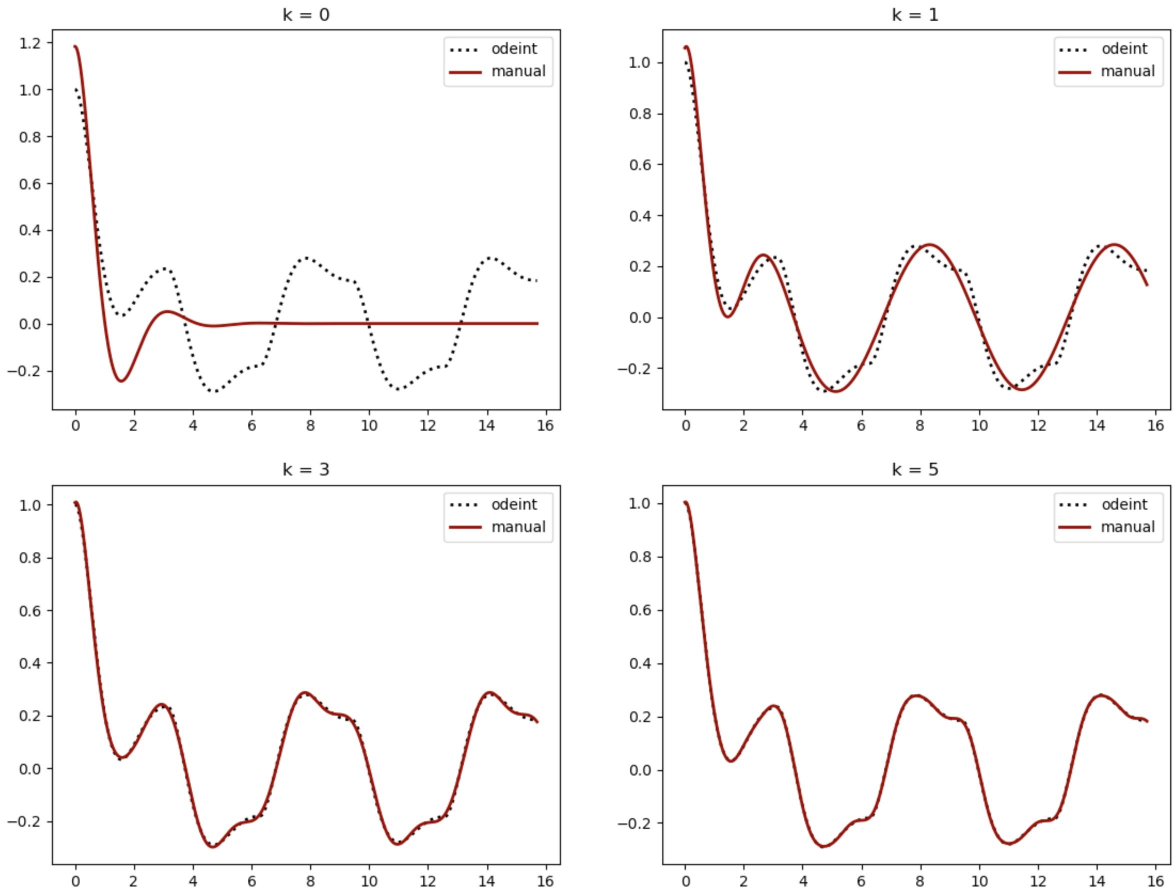


Figure 2: Solution to $y'' + 2y' + 5y = f_{sq}$ with $y(0) = 1$ and $y'(0) = 0$. The solution given by what was found in section 2 is labeled 'manual' and the dotted line is the numerical solution of the DE from the scipy.integrate Python library. The value of k above each subplot refers to the k in $y_p(x, k)$.

When $k = 0$, only y_h is plotted (see Figure 2), and we see that by the time $k = 5$, the two solutions are nearly identical. Recall that the even terms do not contribute anything to $y_p(x, k)$ because of the definition of b_n; that is why only odd k are plotted.

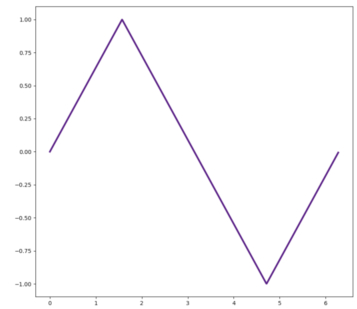
3 Continuous Example: Triangle Wave as Forcing Function

Consider the differential equation

$$y'' + y' + 16y = f_{tr}$$

where

$$f_{tr}(x) = \begin{cases} \frac{2}{\pi}x & \text{for } x < \frac{\pi}{2} \pmod{2\pi} \\ \frac{-2}{\pi}(x-\pi) & \text{for } \frac{\pi}{2} < x < \frac{3\pi}{2} \pmod{2\pi} \\ \frac{2}{\pi}(x-2\pi) & \text{for } \frac{3\pi}{2} < x \pmod{2\pi} \end{cases}$$



and initial conditions

$$y(0) = \frac{1}{2}, \quad y'(0) = 0$$

3.1 Computing Fourier Series of f_{tr}

Since the triangle wave is an odd function, its coefficients for the cos terms, a_0 and a_n, are zero. So we need only compute the coefficients for b_n.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} f_{tr}(x)\sin(nx)dx \\ &= \frac{1}{\pi}\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x-\pi)\sin(nx)dx + \int_{\frac{3\pi}{2}}^{2\pi} (x-2\pi)\sin(nx)dx\right] \end{aligned}$$

Recall that by using integration by parts,

$$\int udv = uv - \int vdu.$$

Let us find $\int_a^b (x-\lambda)\sin(nx)dx$ and use that solution for each of the three integrals.

$$u = x - \lambda \qquad\qquad du = 1$$

$$v = \frac{-1}{n}\cos(nx) \qquad\qquad dv = \sin(nx)$$

$$\begin{aligned} \int_a^b (x-\lambda)\sin(nx)dx &= \left[(x-\lambda)\frac{-1}{n}\cos(nx)\right]\Bigg|_a^b - \int_a^b \frac{-1}{n}\cos(nx) \\ &= \frac{1}{n}\left[(a-\lambda)\cos(na) - (b-\lambda)\cos(nb) + \frac{1}{n}\Big(\sin(nb) - \sin(na)\Big)\right] \\ &= \frac{n(a-\lambda)\cos(na) - \sin(na) + \sin(nb) - n(b-\lambda)\cos(nb)}{n^2} \end{aligned}$$

Applying this to find b_n, we get

$$b_n = \frac{2}{\pi^2}\left[\int_0^{\frac{\pi}{2}} x\sin(nx)dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x-\pi)\sin(nx)dx + \int_{\frac{3\pi}{2}}^{2\pi} (x-2\pi)\sin(nx)dx\right]$$

$$= \frac{2}{\pi^2}\left[\frac{n(0-0)\cos(n0) - \sin(n0) + \sin(n\frac{\pi}{2}) - n(\frac{\pi}{2}-0)\cos(n\frac{\pi}{2})}{n^2}\right.$$

$$- \frac{n(\frac{\pi}{2}-\pi)\cos(n\frac{\pi}{2}) - \sin(n\frac{\pi}{2}) + \sin(n\frac{3\pi}{2}) - n(\frac{3\pi}{2}-\pi)\cos(n\frac{3\pi}{2})}{n^2}$$

$$\left. + \frac{n(\frac{3\pi}{2}-2\pi)\cos(n\frac{3\pi}{2}) - \sin(n\frac{3\pi}{2}) + \sin(n2\pi) - n(2\pi-2\pi)\cos(n2\pi)}{n^2}\right]$$

$$= \frac{2}{n^2\pi^2}\left[\sin(n\frac{\pi}{2}) - \cancel{\frac{n\pi}{2}\cos(n\frac{\pi}{2})}\right.$$

$$- \left(\cancel{\frac{-n\pi}{2}\cos(n\frac{\pi}{2})} - \sin(n\frac{\pi}{2}) + \sin(n\frac{3\pi}{2}) - \cancel{\frac{n\pi}{2}\cos(n\frac{3\pi}{2})}\right)$$

$$\left. + \cancel{\frac{-n\pi}{2}\cos(n\frac{3\pi}{2})} - \sin(n\frac{3\pi}{2})\right]$$

$$= \frac{2}{n^2\pi^2}\left[\sin(n\frac{\pi}{2}) + \sin(\frac{\pi}{2}) - \sin(n\frac{3\pi}{2}) - \sin(n\frac{3\pi}{2})\right]$$

$$= \frac{4}{n^2\pi^2}\left[\sin(n\frac{\pi}{2}) - \sin(n\frac{3\pi}{2})\right]$$

Observing that

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 1 & \text{for } n \equiv 1 \pmod 4 \\ 0 & \text{for } n \equiv 2 \pmod 4 \\ -1 & \text{for } n \equiv 3 \pmod 4 \\ 0 & \text{for } n \equiv 0 \pmod 4 \end{cases} \qquad \sin\left(n\frac{3\pi}{2}\right) = \begin{cases} -1 & \text{for } n \equiv 1 \pmod 4 \\ 0 & \text{for } n \equiv 2 \pmod 4 \\ 1 & \text{for } n \equiv 3 \pmod 4 \\ 0 & \text{for } n \equiv 0 \pmod 4 \end{cases}$$

Therefore

$$b_n = \begin{cases} \frac{8}{n^2\pi^2} & \text{for } n \equiv 1 \pmod 4 \\ \frac{-8}{n^2\pi^2} & \text{for } n \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Or equivalently,

$$b_n = \frac{8(-1)^{\frac{(n-1)}{2}}}{n^2\pi^2} \text{ for } n = 1, 3, 5, \ldots, \text{ otherwise } 0$$

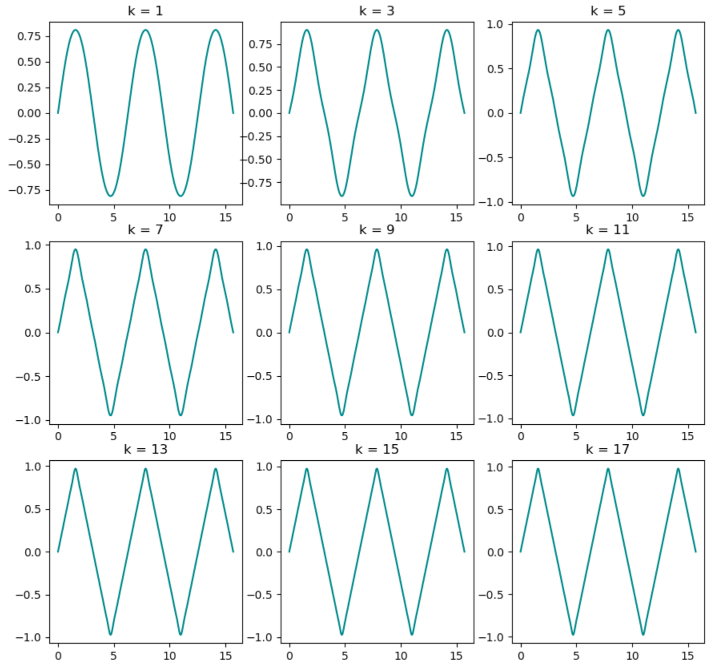


Figure 3: The Fourier series of f_{tr} for finitely many terms k, i.e., each subplot shows $\sum_{n=1}^{k} b_n \sin(nx)$.

3.2 Computing y_p

Recall from section 2.3 that if we have a differential equation

$$y'' + Ey' + Fy = \sum_{n=1}^{\infty} b_n \sin(nx)$$

then the values of $a_n^\star$ and $b_n^\star$ for

$$y_p(x) = \sum_{n=1}^{\infty} a_n^\star \cos(nx) + \sum_{n=1}^{\infty} b_n^\star \sin(nx)$$

are

$$a_n^\star = \frac{-Enb_n}{(F-n^2)^2 + E^2n^2}, \quad b_n^\star = \frac{(F-n^2)b_n}{(F-n^2)^2 + E^2n^2}.$$

Since we are now interested in the differential equation

$$y'' + y' + 16y = f_{tr},$$

we have that

$$a_n^\star = \frac{-nb_n}{(16-n^2)^2 + n^2}, \qquad b_n^\star = \frac{(16-n^2)b_n}{(16-n^2)^2 + n^2}$$

where b_n is just as computed in the previous section.

3.3 Computing y_h

$$\begin{aligned} r &= \frac{-1 \pm \sqrt{1-64}}{2} \\ &= \frac{-1}{2} \pm \frac{3}{2}\sqrt{7}i \end{aligned}$$

Therefore

$$y_h(x) = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{3}{2}\sqrt{7}x\right) + c_2 \sin\left(\frac{3}{2}\sqrt{7}x\right)\right)$$

3.4 Initial Conditions

Given that

$$y(0) = \frac{1}{2}, \quad y'(0) = 0,$$

we find

$$\begin{aligned} y(0) &= \sum_{n=1}^{\infty} a_n^\star \cos(n0) + \sum_{n=1}^{\infty} b_n^\star \sin(n0) + e^{-\frac{1}{2}0}\left(c_1 \cos\left(\frac{3}{2}\sqrt{7}\cdot 0\right) + c_2 \sin\left(\frac{3}{2}\sqrt{7}\cdot 0\right)\right) \\ &= \sum_{n=1}^{\infty} a_n^\star + c_1 = \frac{1}{2} \implies c_1 = \frac{1}{2} - \sum_{n=1}^{\infty} a_n^\star \end{aligned}$$

$$\begin{aligned} y'(0) = &\sum_{n=1}^{\infty} -na_n^\star \sin(n0) + \sum_{n=1}^{\infty} nb_n^\star \cos(n0) - \frac{1}{2}e^{-\frac{1}{2}0}\left(c_1 \cos\left(\frac{3}{2}\sqrt{7}\cdot 0\right) + c_2 \sin\left(\frac{3}{2}\sqrt{7}\cdot 0\right)\right) + \\ &e^{-\frac{1}{2}0}\left(-c_1\frac{3}{2}\sqrt{7}\sin\left(\frac{3}{2}\sqrt{7}\cdot 0\right) + c_2\frac{3}{2}\sqrt{7}\cos\left(\frac{3}{2}\sqrt{7}\cdot 0\right)\right) \\ = &\sum_{n=1}^{\infty} nb_n^\star - \frac{1}{2}c_1 + \frac{3}{2}\sqrt{7}c_2 = 0 \\ &\implies c_2 = \frac{2}{3\sqrt{7}}\left(\frac{c_1}{2} - \sum_{n=1}^{\infty} nb_n^\star\right) \end{aligned}$$

3.5 Plotting Solutions of $y'' + y' + 16y = f_{tr}$

With c_1 and c_2 found, we have everything to plot the solution to the DE $y'' + y' + 16 = f_{tr}$.

$$\begin{aligned} y(x) &= y_p(x,k) + y_h(x) \\ &= \sum_{n=1}^{k} a_n^\star \cos(nx) + \sum_{n=1}^{k} b_n^\star \sin(nx) + e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{3}{2}\sqrt{7}x\right) + c_2 \sin\left(\frac{3}{2}\sqrt{7}x\right)\right) \end{aligned}$$

where

$$a_n^\star = \frac{-nb_n}{(16-n^2)^2 + n^2}, \qquad b_n^\star = \frac{(16-n^2)b_n}{(16-n^2)^2 + n^2}$$

and

$$c_1 = \frac{1}{2} - \sum_{n=1}^{\infty} a_n^\star, \qquad c_2 = \frac{2}{3\sqrt{7}}\left(\frac{c_1}{2} - \sum_{n=1}^{\infty} nb_n^\star\right)$$

This time,

$$\sum_{n=1}^{10000} a_n^\star \approx -3.719 \times 10^{-4} \qquad \text{and} \qquad \sum_{n=1}^{10000} nb_n^\star \approx 9.867 \times 10^{-3}.$$

And they both ostensibly converge to this value for arbitrarily large k.

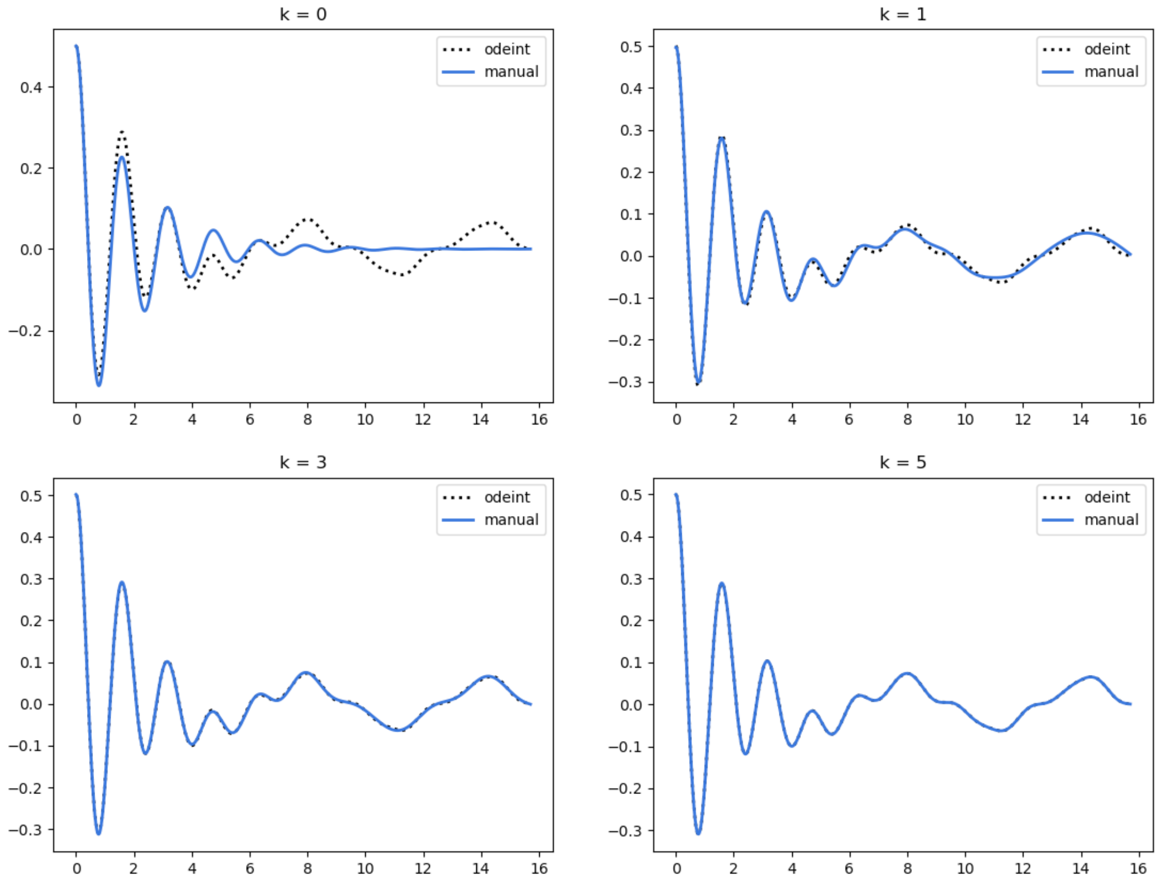


Figure 4: Solution to $y'' + y' + 16y = f_{tr}$ with $y(0) = \frac{1}{2}$ and $y'(0) = 0$. The solution given by what was found in section 3 is labeled 'manual' and the dotted line is the numerical solution of the DE using the scipy.integrate Python library. The value of k above each subplot refers to the k in $y_p(x, k)$.

Of course, the solutions match as we consider larger k for $y_p(x, k)$, though it resembles to the numerical solution quite quickly; after only taking $k = 5$ terms of y_p the two are nearly indistinguishable, similar to the first example with the discontinuous forcing function.

4 Conclusion

While some of the computations might have looked lengthy, the overarching idea for how one solves these ODEs using Fourier series is rather simple. All we need to do is find the Fourier series of the forcing function, then the rest is just algebraic manipulation. Surprisingly too, for the examples we considered, we don't need to evaluate so many terms of y_p (i.e., we do not need even moderately large k) before

$$y(x) \sim y_p(x, k) + y_h(x).$$